ELK ASIA PACIFIC JOURNAL OF MECHANICAL ENGINEERING RESEARCH

ISSN Online: 2394-9368 Volume 2 Issue 1 (2016)

ELK
Asia Pacific Journals

www.elkjournals.com

NONLINEAR DIFFUSION-REACTION EQUATIONS WITH NONLINEAR CONVECTIVE FLUX TERM

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ABSTRACT

Attempts have been made to look for the soliton content in the exact solutions of certain types of nonlinear diffusion-reaction (D-R) equations which involve not only the quadratic and quartic nonlinearities but also a nonlinear Convective flux term. Such equations may arise in a variety of contexts in physical problems. In particular, the kink and antikink shaped soliton solutions are found. It is noticed that in mathematical terms a parallel D-R equation with cubic nonlinearity and a constant-velocity-convective term can take over the effects of both quadratic and cubic nonlinearities in the convective flux term in the above D-R equations.

Index Terms—Nonlinear diffusion equation, Convective flux term, Soliton

INTRODUCTION

The role of nonlinearities in the studies of many phenomena in nature starts manifesting as and when one thinks of departing from the 'idealized linear situation' in the process of modelling the phenomenon under study. This generally leads to more realistic studies and brings the theory closer to the refined experimental data. Sometimes the proper choice of the nonlinearity in the space-time evolution equation of the system itself becomes a problem mainly because of its plurality in mathematical terms. Even if it is appropriately chosen, then the next stage of difficulty arises in obtaining an exact analytical solution of the concerned differential equation. For the solutions of these equations, normally one resorts to approximation or numerical methods which again have certain limitations. In fact the exact solution of a nonlinear differential equation, if becomes available, has its own beauty in the sense that one can easily study the dependence of the solution on the

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underlying (physical) parameters analytically which otherwise is lost in the approximation or numerical methods. The purpose of this letter is to explore the soliton content in the solutions of certain types of nonlinear diffusion-reaction (D-R) equations which involve quadratic and quartic nonlinearities with a nonlinear 'convective flux term'. In particular, we investigate the exact solutions of the nonlinear D-R equations,

$$C_t + kCC_x = DC_{xx} + aC - bC^2, \quad (1)$$

And $C_t + kC^2C_x = DC_{xx} + \alpha C - bC^4$, (2)

where C = $C(x,t)$, is the concentration or the density variable depending on the phenomenon under study; D is the diffusion coefficient, and k, a, b are real constants. With regard to the physical and mathematical content of Eqs. (1) and (2) the following remarks are in the order:

(i) The motivation to study these equations comes from some recent works [1-6], particularly those of Moiseyeb and Gluck [2] and that of Nelson and Snerb [3, 4] in the field of population biology. In fact, the study of these equations could be of immense use in a variety of fields where they mainly arises in analogous forms [7].

(ii) Equations (1) and (2) describe a transport phenomenon in which both diffusion and convection processes are of equal importance, i.e., the nonlinear diffusion could be thought of as equivalent to the nonlinear convection effects. In particular, the second term on the left hand side, $kCC_x (or\ kC^2C_x)$ is the replacement of the conventional uC_{x^-} term [2], where v can be a function of both x and t in general but it is considered as a constant in most of the earlier works [3, 4, 8, 9]. Thus, by choosing this term as $kCC_x (or\ kC^2C_x)$ in the present work, we are considering higher order generalization in the spirit of point (iii) below.

(iii) It may be mentioned that the presence of uC_{x^-} type convective flux term in the D-R equation [8] makes the system non-conservative, whereas a nonlinear convection term in (1) or in (2) arise as a natural extension of a conservation law [1]. This can be demonstrated by writing the D-R equation in the case of density dependent models as [1]

$$C_t + h_x(C) = DC_{xx} + f(C), \quad (3)$$

Where h(C) is some function of C. In Eq.

(3), the choices of $f(C) = \alpha C - bC^2$ and of

$$h_x(C) = (\frac{\partial h(C)}{\partial C})C_x = kCC_x(or\ kC^2C_2)$$

While 2

lead to Eq. (1) (or Eq.(2)), the L.H.S. of (3) is expressible in the form of a divergence, namely $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x})(C, h(C))$.The 'convective velocity' now becomes $\frac{\partial h}{\partial C}$ and the same is chosen here as $kC(or\ kC^2)$ with k a positive or negative constant.

(iv) No doubt, the form (3) of Eq. (1) has been studied [1] in a variety of situations like in the studies of ion-exchange columns, chromatography etc., but Eq. (2), which we are investigating here perhaps for the first time, could also be a potential candidate from this point of view for such studies. Further note that the presence of a nonlinear convection term in the D-R equations can lead to a dramatic effect on their solutions.

2 EXACT SOLUTIONS OF EQS. (1) AND

(2) Now, by defining a variable $\xi = x - wt$, Eqs. (1) and (2) can be expressed respectively as

$$DC'' + \omega C' - kCC' + \alpha C - bC^2 = 0,$$

(4)

$$DC'' + \omega C' - kC^2C' + \alpha C - bC^4 = 0,$$

(5)

For the solution of (4) we make an ansatz [9]

$$C(\xi) = a_0 + a_1 \tanh(\mu\xi),$$

(6)

where a0; a1; ; in addition to w are the arbitrary constants to be determined later. In fact the use of Eq. (6) in (4) and the rationalization of the resultant expression with respect to the powers of tanh( ) will yield the following set of equations:

$$\alpha a_0^2 + \omega a_1\mu - ka_0a_1\mu = 0,$$

(7)

$$\alpha a_1 - 2a_1D\mu^2 - 2ba_0a_1 - ka_1^2\mu = 0,$$

(8)

$$-ba_2^2 - wa_1\mu + ka_0a_1\mu = 0,$$

(9)

$$2a_1a_0\mu^2 + ka_1^2\mu = 0,$$

(10)

which can be solved for the four unknowns $a_0, a_1\mu$ a0; a1; and w to give

$$a_0 = \frac{\alpha}{2b}; = \mp\frac{\alpha}{2b}; \mu = \pm\frac{k\alpha}{4Db};$$

$$w = \frac{k^2\alpha + 4Db^2}{2bk}.$$

(11)

(Refer Fig.1) and finally, the solution $C(\xi)$

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of Eq. (4) turns out to be

$$C(\xi) = \frac{\alpha}{2b}\left[1 \mp \tanh\left(\pm\frac{k\alpha}{4Db}\xi\right)\right]. \quad (12)$$

As in our earlier work [9], we denote these solutions as C_{-+} and C_{++} corresponding to upper and lower signs in (12) for the positive k and as C and C_{++} for the negative k. In fact it turns out that while positive values of k provide the kink soliton solutions, the negative values of k, on the other hand, lead to antikink soliton solutions. Further since $\tanh(\mu\xi)$ is an odd function of ξ therefore note that $C_{-+} = C_{+-}$ and $C_{--} = C_{++}$.

Similarly if one uses the ansatz (6) in (5) and rationalizes the resultant expression with respect to the powers of tanh( ), one obtains a set of five equations which again can be solved for the unknowns a_0; a_1; and w to give

$$a_0 = \frac{bD}{k^2}; = \pm\frac{bD}{k^2}; \mu = \pm\frac{b^2D}{k^3}$$

$$w = \frac{k^6\alpha + 16b^4D^3}{4b^2Dk^3} \quad (13)$$

along with a constraining relation,

$$\alpha = \frac{8b^4D^3}{k^6}$$

among the constant coefficients in (5). Finally the solution of (5) can be written as

$$C(\xi) = \frac{bD}{k^2}\left[1 \pm \frac{b^2D}{k^3}\xi\right] \quad (14)$$

Here, note the negative values of w for positive k contrary to that in Eq. (11). However, the dependence of $C(\xi)$ on the sign of k remains the same for Eqs. (12) and (14) and so are the symmetries of the solutions C_{++}, C_{--}, C_{++} and C_{-+}

EXISTENCE OF KINK AND ANTIKINK SHAPED SOLITONS

An interesting aspect of solutions (12) and (14), respectively of Eqs. (1) and (2) is that they correspond to the solution of the D-R equation with cubic nonlinearity but with a constant-velocity-convective term, viz.

$$C_t + uC_x = DC_{xx} + \alpha C - bC^3. \quad (15)$$

Fig. 1. Solutions of Eq. (4): (a) Kink-shaped soliton (C_{++}) and
(b) antikink shaped soliton (C_{+}), for D = 0:02; = 0:7; b = 0:8.

Equation (15) is found [9] to admit the solutions in the form (6) but with the values of $a_0, a_{1,} \mu$

and w now as

$$a_0 = a_1 = \pm\left(\frac{\alpha}{2b}\right), \mu = \pm\sqrt{\left(\frac{\alpha}{8D}\right)}$$

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and $v - w = \pm 3\sqrt{\left(\frac{\alpha D}{2}\right)}$.

the three nonlinear D-R Eqs. (1), (2) and (15) are found to have the same mathematical structure, viz. Eq. (6), it may be argued that the effect of nonlinear convective terms in Eqs. (1) and (2) is taken over by a constant-velocity-convective term in (15). However, the solutions might have different features in numerical terms (cf. Figs. 1- 3).

The soliton solutions of Eqs. (1) and (2) are shown in Figs. 1 and 2 as a function of and for different positive values of ξ and for $D = 0.02, b = 0.2$. Note that while = 0:7 is used in Fig. 1, the value of is however derived from the constraint, $\alpha = \frac{8b^4D^3}{k^6}$ in the

solutions **(Refer Fig.2)**

(5): (a) kink-shaped soliton (C_{++}) and (b) antikink shaped soliton (C_{+}), for D = 0:02; b = 0:8. for Eq. (5). In Fig. 3, we display the solutions of Eq. (15) for different values of v mainly for sake of comparison. It can be seen that the steepness of the kink and antikink solitons decreases with the values of k for (5), (cf. Fig. 2) whereas it remains more or less unaffected with the values of k for Eq. (4) and also value of v for Eq. (15) (cf. Figs. 1 and 3).

In fact these features of the solutions conform to the observations made in Ref. [1] (cf. p. 288), i.e., higher the order of nonlinearity in a D-R equation lesser is the steepness of kink or antikink. If one also accounts for the linear $vC_x - type$ convective term in Eqs. (1) and (2) along with the corresponding nonlinear convective terms, then w in the solutions (12) and (14) will now be replaced by $(w - v)$. This leads to an enhanced effective value of ξ for a given x and t, of course for positive values of w and for w > v. Another

Fig. 3. Solution of Eq. (15) for: (a) kink-shaped soliton (C_{++}) and (b) antikink shaped soliton (C_{+}), D = 0:02; = 0:7; b = 0:8. aspect to be noted in the solutions of (4) and

(5) is that the later involves rather complicated dependence on the parameter k (cf. Eq. (13)). Not only this, solitary wave velocity w turns out to have opposite signs in the two cases for positive k. This implies that not only the kink solution of Eq. (4) is analogous to the antikink solution of Eq. (6) but also the associated waves travel in opposite directions. Similar conclusion can be drawn for the negative values of k vis-a-vis the algebraic signs in Eqs. (12) and (14).

4. CONCLUDING REMARKS

With a view to extending the scope of

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applications of nonlinear D-R equations with quadratic and quartic nonlinearities, the role of certain types of nonlinear convective terms in these equations is investigated. The existence of the kink and antikink shaped soliton solutions is demonstrated in certain parametric domain. From this point of view of further application it may be of interest to study the time de-pendence of the parameters appearing in these equations. Such studies are in progress.

ACKNOWLEDGMENTS

We would like to thank R. S. Kaushal and Awadhesh Prasad for the helpful discussion.

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LIST OF FIGURES:

FIG.1

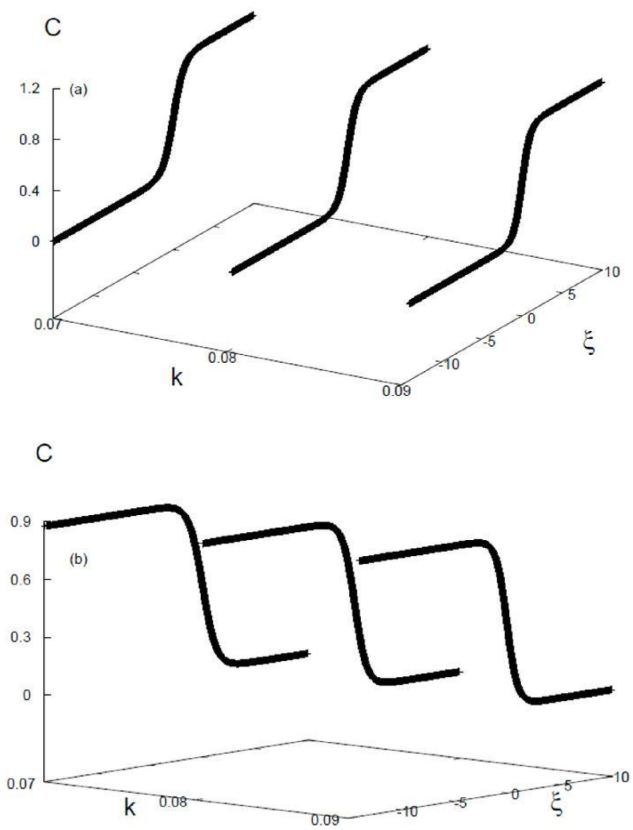

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FIG. 2. SOLUTIONS OF EQ